

Torus Fractalization and Intermittency

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Abstract

The bifurcation transition is studied for the onset of intermittency analogous to the Pomeau-Manneville mechanism of type-I, but generalized for the presence of a quasiperiodic external force. The analysis is concentrated on the torus-fractalization (TF) critical point that occurs at some critical amplitude of driving. (At smaller amplitudes the bifurcation corresponds to a collision and subsequent disappearance of two smooth invariant curves, and at larger amplitudes it is a touch of attractor and repeller at some fractal set without coincidence.) For the TF critical point, renormalization group (RG) analysis is developed. For the golden mean rotation number a nontrivial fixed-point solution of the RG equation is found in a class of fractional-linear functions with coefficients depending on the phase variable. Universal constants are computed responsible for scaling in phase space ($\alpha = 2.890053\dots$ and $\beta = -1.618034\dots$) and in parameter space ($\delta_1 = 3.134272\dots$ and $\delta_2 = 1.618034\dots$). An analogy with the Harper equation is outlined, which reveals important peculiarities of the transition. For amplitudes of driving less than the critical value the transition leads (in the presence of an appropriate re-injection mechanism) to intermittent chaotic regimes; in the supercritical case it gives rise to a strange nonchaotic attractor.

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1 Introduction

It is commonly believed that under parameter variation the turbulent dynamics in multi-dimensional systems may arise via quasiperiodicity, in a course of subsequent birth of oscillatory components with incommensurate frequencies, followed by chaotization (see e.g., the works of Landau, Hopf, and Ruelle and Takens [1,2,3]).

Now it is well known that actual details of the transition from quasiperiodicity to chaos are very subtle and complicated. Some of them can be revealed if we turn to a kind of restricted problem: Suppose that the system may be decomposed to a master subsystem with quasiperiodic behavior and a driven slave subsystem, and the last can demonstrate transition to chaos. So, we may ask what are possible scenarios of the onset of chaos in the second subsystem? (Note an analogy with an approach to the problem of three bodies in celestial mechanics: Being difficult in a general formulation, it allows essential advance in a restricted version, under a suggestion that one of the bodies is of negligible mass.) One of the important results on this way of reasoning was a formulation of the concept of strange nonchaotic attractor (SNA), which typically appears in an intermediate region between order and chaos [4-6]. In the phase space this is an object of fractal geometrical structure, but without instability in respect to the initial conditions in the driven system.

One more essential idea consists in application of the renormalization group (RG) approach, proven to be very efficient for understanding dynamics in critical states between order and chaos (e.g. [7-16]). Starting with an evolution operator for some definite time interval we are able to construct the evolution operator for a larger interval. Then, we try to produce an appropriate variable change to make the new operator as close as possible to the original. This is just one step of the RG transformation, and it may be repeated again and again to obtain operators for larger and larger time intervals. As a result, we arrive at some universal operator, which describes long-time evolution of the system at the criticality. It is often represented by a fixed point of the RG transformation. Studies of this fixed-point operator together with consideration of its relevant perturbations reveal properties of universality and scaling for the transition. Originally, such an approach was developed by Feigenbaum for the period-doubling scenario of the onset of chaos [7,8]; afterwards it was applied

to many other situations, including quasiperiodicity at the chaos border [10-12] and some cases of the birth of SNA [14-16].

Finally, we have to mention here a concept of intermittency suggested by Pomeau and Manneville [17]. It occurs in very general circumstances near a saddle-node bifurcation (also called 'tangent bifurcation', preferably in the context of 1D maps). It has been studied in different aspects by many authors. In particular, the RG approach has been applied to intermittency in Refs. [18,19].

The goal of the present article is to consider a generalization of the Pomeau-Manneville mechanism for the case of the presence of quasiperiodic driving, and to reveal details of the bifurcation transition, which is an analog of the tangent bifurcation in this case. It is natural to regard the situation as one of possible scenarios of transition from quasiperiodicity to chaos in the context of the mentioned "restricted problem". At small amplitudes of driving, the transition is rather trivial and consists in collision with coincidence (and subsequent disappearance) of a pair of smooth stable and unstable tori, see e.g. [20, 21]. However, at a definite value of the amplitude a non-trivial critical situation occurs. It allows application of the RG approach that will be developed. Also, the associated scaling properties will be revealed and discussed.

In Sec.II we introduce the basic model map and review its dynamical phenomena in the presence of the external quasiperiodic driving. In Sec.III we consider some details of dynamics in terms of rational approximations of the frequency parameter and locate numerically the critical point associated with threshold of fractalization at the moment of collision of the invariant curves. In Sec.IV we discuss a link between the problem under study and the Harper equation — the lattice version of the one-dimensional Schrödinger equation, well known in the context of solid-state physics [22-27]. In Sec.V the RG analysis is developed for the situation of tori fractalization: The RG equation is derived, and results of its numerical solution are presented. In Sec.VI we discuss scaling properties of the dynamics at the critical point. In Sec.VII the linearized RG equation is derived, the spectrum of eigenvalues is obtained, and two relevant eigenvalues responsible for scaling in the parameter plane are distinguished. In Sec.VIII we consider consequences of these results concerning dynamics in a neighborhood of the critical point in the parameter space. In particular, we extract from the RG results the critical exponents for the duration of the laminar stages of intermittency and compare them with empirical numerical data. In conclusion we discuss some perspectives of further studies in the context of the general problem of understanding the transition from usual quasiperiodic regimes ("smooth torus") to SNA and chaos.

2 The model and basic phenomena

Let us start with an example of quasiperiodically forced 1D map

$$x_{n+1} = f(x_n) + b + \epsilon \cos 2\pi n w, \quad (1)$$

where ϵ and w are the amplitude and frequency parameters of the external force, respectively. We assume that the frequency parameter, called also the rotation number, is taken to be equal to the inverse golden mean, $w = (\sqrt{5} - 1)/2$. As to the function $f(x)$, let us define it here as

$$f(x) = \begin{cases} x/(1-x), & x \leq 0.75, \\ 9/2x - 3, & x > 0.75. \end{cases} \quad (2)$$

(One branch of the mapping is selected in a form of the fractional-linear function, $x/(1-x)$, which appears naturally in analysis of dynamics near the tangent bifurcation associated with intermittency, see e.g. [18,19]. The other branch is attached somewhat arbitrarily to ensure presence of the 're-injection mechanism' in the dynamics.)

At zero amplitude of driving what we have is a usual transition to chaos via the Pomeau-Manneville intermittency of type I, controlled by parameter b , see Fig.1(a). At $b < 0$ the map has two fixed points on the left branch, one stable and one unstable. Under increase of b these points approach one to another, collide, and then disappear. After that, at $b > 0$, the narrow "channel" remains at the place of former existence of the pair of the fixed points, and travel across this channel is a slow process – the laminar stage of intermittency. Closer to the bifurcation point $b = 0$, larger the number of iterations required to pass the channel. After visiting the right-hand branch (the turbulent stage of the intermittency) the orbit quickly returns to the left, and travels through the channel again and again.

If the amplitude of driving is finite (although sufficiently small), then, instead of the fixed points we observe a pair of closed smooth invariant curves, attractor and repeller, see Fig.1(b). (A closed invariant curve may be thought as a cross-section of a torus. For brevity, it is convenient sometimes to speak about stable and unstable tori rather than about the invariant curves.) With increase of b attractor and repeller come nearer to each other and collide, and the localized attractor-repeller pair disappears. After that an extended attractor arises of a form shown in the right panel of Fig.1(b). On the diagram a degree of darkness reflects relative duration of presence of the orbit in different parts of the attractor.

While we remain close to the point of bifurcation, the laminar stages of dynamics may be distinguished, which occupy an overwhelming part of observation time, like in the case of the usual Pomeau-Manneville intermittency. In Fig.1(b) they correspond to a domain of the most long-living residence – along the left branch of the map, at the place of location of the former attractor-repeller pair. In our study we will concentrate on the analysis of the laminar stages in the same way as it is commonly accepted in the case of conventional intermittency. For this, it is sufficient to use the map

$$x_{n+1} = x_n/(1 - x_n) + b + \epsilon \cos(2\pi n w). \quad (3)$$

As the numerical simulations clearly demonstrate, the collision of the attractor-repeller pair is of different nature at small and at large amplitudes of driving. Similar observations were reported earlier in computations for the driven circle map [21,28].

At ϵ less than some critical value ϵ_c (in our map $\epsilon_c = 2$) we observe that the invariant curves remain smooth until the collision, and they precisely coincide with one another at that moment (Fig.2(a)). For $\epsilon = \epsilon_c$ they also coincide at the collision, but here the form of the invariant curve appears to be wrinkled (Fig.2(b)). Finally, at $\epsilon > \epsilon_c$, one can see that the collision takes place only at some fractal subset of points on the invariant curves, and no coincidence of the entire curves is observed (Fig.2(c)).

The essential change in the nature of the transition with passage from $\epsilon < \epsilon_c$ to $\epsilon > \epsilon_c$ may be demonstrated also by computations of the Lyapunov exponent. Figure 3(a) shows the Lyapunov exponent as it behaves along the bifurcation border (at the collision of the attractor-repeller pair) as a function of the amplitude of driving. Observe that for $\epsilon < \epsilon_c$ the Lyapunov exponent has constant zero value, but for $\epsilon > \epsilon_c$ it becomes negative and decreases with growth of ϵ according to a visually perfect linear law. Figure 3(b) depicts a diagram for the Lyapunov exponent dependence on b at fixed $\epsilon = \epsilon_c$; it shows essentially distinct behavior, apparently, a power law with a non-trivial exponent.

The intriguing fact that the change of character of the bifurcation in the model map (3) takes place precisely at $\epsilon_c = 2$ will be explained in Section IV.

3 Method of rational approximations

As is well known, the irrational $w = (\sqrt{5}-1)/2$ taken as the frequency parameter in the driven map is a limit of a sequence of rationals $w_k = F_{k-1}/F_k$, where F_k are the Fibonacci numbers ($F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$). Let us change the rotational number w to its approximant w_k , and introduce a parameter of initial phase u into the equation:

$$x_{n+1} = x_n/(1 - x_n) + b + \epsilon \cos 2\pi(n w_k + u). \quad (4)$$

Now, below the transition, at any fixed u we have a pair of cycles of period F_k , one stable, and another unstable. The Floquet eigenvalue, or multiplier μ , which characterizes decrease of a perturbation over one period of the stable cycle, will depend on u . This dependence appears to be periodic, with period $2\pi/F_k$. For a given ϵ we may select numerically such b that the maximal value of μ at some phase reaches 1, and μ is less than 1 at other phases. It corresponds just to the first tangent bifurcation, that is a collision of the earlier stable cycle of period F_k with its unstable partner. Technically, the computations are simplified with two observations: first, the maximum of the multiplier occurs at $u = 0$, and, second, the initial condition for the cycle at the situation of collision may be expressed explicitly (see Eq. (16) in Sec.IV).

In Table I we present numerical data for the values of b corresponding to the cycle collision at $u = 0$ for the critical amplitude $\epsilon = 2$. Figure 4 shows the multiplier as a function of the phase u in an interval of periodicity in a moment of the first tangent bifurcation. The diagrams are plotted at three subsequent levels of the rational approximation for the amplitude parameter less, equal, and larger than 2.

For $\epsilon < 2$ the dependencies become more flat under increase of the order of the rational approximation (the bifurcation tends to become "phase independent"). In contrast, for $\epsilon > 2$ the curves tend to become sharper. At $\epsilon = 2$ the form of the curves looks like stabilized at subsequent levels of the rational approximation.

It is interesting to discuss a relation of these observations with the behavior of the Lyapunov exponent at the transition. As we consider attractor for the irrational rotation number in terms of a certain rational approximant $w_k = F_{k-1}/F_k$, it looks like a collection (continuum set) of periodic orbits, each of which is associated with a particular initial phase u and has a value of Lyapunov exponent $\Lambda(u) = (1/F_k) \ln \mu(u)$. To obtain in this approximation an estimate for the Lyapunov exponent of the whole attractor, we have to perform averaging over the initial phases, $\Lambda = \langle \Lambda(u) \rangle$. From the behavior of the multipliers in the subcritical case we conclude that the value of Λ will tend to zero under increase of the order of rational approximation. In the supercritical case only a very pure subset of the orbits will have multipliers distant from 0 and close to 1, so, the average value of Λ is negative. These arguments are in agreement with the observed dependence of the Lyapunov exponent on the parameter b along the bifurcation curve (see the previous section, Fig.3a).

As seen from Table I, the bifurcation sequence converges to a well-defined limit,

$$b = b_c = -0.597\ 515\ 185\ 376\ 121 \dots \quad (5)$$

(See also a remark in the final part of Sec.VI.) It is a numerical estimate for the parameter value associated with the critical point of fractalization of the colliding tori. It will be referred to as the *TF critical point*. Dynamics at this point and in its vicinity is a main subject of our study in Sec.V and further.

4 A link with Harper equation

A Schrödinger equation in a normalized form for a quantum particle in a one-dimensional discrete lattice with additional quasiperiodic potential reads

$$i \frac{\partial \psi_n}{\partial t} = \psi_{n+1} + \psi_{n-1} - 2\psi_n + (\epsilon \cos 2\pi n w) \psi_n, \quad (6)$$

where n is the spatial index, ϵ is an amplitude of the quasiperiodic potential, and w defines its wave-number. Alternatively, one can speak of a wave process in a lattice medium with supplied sustained quasiperiodic perturbation. For an oscillatory solution of frequency Ω (that corresponds to a state of quantum particle of definite energy) the exponential substitution $\psi_n \propto \exp(i\Omega t)$ yields

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n + (\Omega + \epsilon \cos 2\pi n w) \psi_n = 0. \quad (7)$$

This is the so-called Harper equation well-known in the context of solid-state physics [22-27].

Let us return to our fractional-linear map (3) and perform a variable change

$$x_n = 1 - \psi_n / \psi_{n-1}. \quad (8)$$

The result is exactly the Harper equation (7) with Ω changed to b :

$$\psi_{n+1} + \psi_{n-1} - 2\psi_n + (b + \epsilon \cos 2\pi n w) \psi_n = 0. \quad (9)$$

The link between the Harper equation and the fractional-linear mappings was noticed and exploited earlier by Ketoja and Satija [25], although they were interested in some other problems than that of our concern here. Recently the same idea was effectively used for analysis of spectral properties of the Harper equation in Ref. [27].

At rational approximants of the wave-number w the expression (7) becomes an equation with periodic coefficients. Together with the Floquet condition

$$\psi_{n+q} = \mu \psi_n = \psi_n e^{i\tilde{\beta}q}, \quad \tilde{\beta} = (\arg \mu + 2\pi m)/q, \quad q = F_k \quad (10)$$

it gives rise to an eigenvalue problem: At any given wave-number $\tilde{\beta}$ one can obtain (say, numerically [24]) a spectrum of frequencies

$$\Omega = \Omega(\tilde{\beta}). \quad (11)$$

It is called a dispersion equation for the waves in the medium governed by Eq.(6). If the equation has a real root $\tilde{\beta}$ at a given Ω , it corresponds to wave propagation, or a transmission band. If the equation has no real, but complex solutions, we say that Ω is in a forbidden zone, or in a non-transmission band. In this case no propagating waves, but spatial exponential decay occurs at the given frequency.

To understand relation between nature of solutions of the Harper equation and those of our original problem, let us turn to a particular case of slow spatial variation of ψ_n , use the continuous limit, and set $\epsilon = 0$. That yields

$$\Psi'' + b\Psi = 0. \quad (12)$$

Now, at $b < 0$ we have solutions of the form $\Psi_n = C \exp(\pm \sqrt{|b|}n)$, and, according to (12), $x_n = 1 - \psi_n/\psi_{n-1} = 1 - \exp(\pm \sqrt{|b|})$. It corresponds to presence of two fixed points. At $b > 0$ we obtain $\Psi_n = C \cos(\sqrt{b}n)$. Then, $x_n = 1 - \psi_n/\psi_{n-1} \rightarrow \infty$ as $n \rightarrow \pi/(2\sqrt{b})$; it means that no localized attractor is present. In the same manner, a forbidden zone of the Harper equation must be associated with existence of a localized attractor-repeller pair of the driven fractional-linear map, while a transmission band corresponds to presence of the "channel" and to the laminar stages of intermittency (see also [27]).

It is easy to find that for $\epsilon = 0$ the transmission band in the Harper model occupies an interval of Ω from 0 to 4. At nonzero ϵ the forbidden zones ('gaps') arise inside the band, and they become wider as ϵ grows. Under increase of the order of rational approximation k , new and new narrower gaps appear inside the transmission bands. It occurs that for $\epsilon < 2$ in the limit $k \rightarrow \infty$ the transmission bands of higher orders dominate over the forbidden zones (i.e. they have a larger total width), but for $\epsilon > 2$ the situation is opposite [23,24]. The transition from one type of behavior to another is known as the localization-delocalization transition. The structure of the transmission bands at the transition appears to be a kind of Cantor-like set. Figure 5(a) shows the transmission and forbidden zones colored, respectively, by gray and white on the parameter plane (ϵ, Ω) .

The fact that the transition in the Harper equation must take place at $\epsilon = 2$ follows from the argument of Aubry [29,24]. By means of the Fourier-like transformation

$$\phi_k = \hat{F}\psi_n = \sum_{n=-\infty}^{\infty} \psi_n e^{2\pi i n k w} \quad (13)$$

one obtains from (7) an equation of similar form

$$\phi_{k+1} + \phi_{k-1} - 2\phi_k + (\Omega' + \epsilon' \cos(2\pi i k w))\phi_k = 0, \quad (14)$$

but with parameters

$$\Omega' = 2 + 2(\Omega - 2)/\epsilon, \quad \epsilon' = 4/\epsilon. \quad (15)$$

Localization of a wave-function implies delocalization of its transform, and vice versa. So, the transition has to occur at $\epsilon = 2$ that corresponds to a fixed point of the equation for ϵ .

In Fig.5(a) one can find a forbidden zone in the bottom part of the diagram. Obviously, its top border must correspond to the threshold of intermittency in the fractional-linear map. As may be observed from comparison of Fig.5(a) and (b), this is indeed the case. The TF critical point found in the previous section corresponds exactly to the lowest frequency associated with appearance of the wave propagation at the localization - delocalization transition in the Harper equation (at $\epsilon = 2$).

The Harper equation (9) possesses an evident symmetry being invariant in respect to the spatial reflection $n \rightarrow -n$. Hence, it is possible to construct a symmetric solution. For this we have to set $\psi_1 = \psi_{-1} = [1 - (b + \epsilon)/2]\psi_0$. According to (8), the respective orbit of the fractional-linear map (3) has an initial condition

$$x_0 = (b + \epsilon)/(b + \epsilon - 2). \quad (16)$$

If this orbit is localized (that occurs at one special value of b for each ϵ), it will correspond to the situation of the attractor-repeller collision. As well, this is true for periodic orbits corresponding to rational approximants w_k at $u = 0$ (see Eq.(4)), and this notion is technically useful for computation of the sequence of parameter b values converging to the critical point TF (Sec.III).

5 Renormalization group analysis

Let us develop now the RG approach to the dynamics at the critical point TF. Here we prefer to write out the original model in a form of two-dimensional mapping

$$\begin{aligned} x_{n+1} &= x_n / (1 - x_n) + b + \epsilon \cos(2\pi u_n), \\ u_{n+1} &= u_n + w \pmod{1}, \end{aligned} \quad (17)$$

and assume, for convenience, that the phase variable u is defined in such way that it always belongs to the interval $(-0.5, 0.5)$.

As the frequency parameter is the inverse golden mean, it is natural to deal with the evolution operators corresponding to Fibonacci's numbers of iterations.

We need to introduce here a new variable X (it differs from x by a u -dependent shift, but details will be explained below). Let $f^{F_k}(X, u)$ and $f^{F_{k+1}}(X, u)$ are the functions representing transformation of X after F_k and F_{k+1} iterations, respectively. To construct the next operator, for F_{k+2} steps, we start from (X, u) and perform first F_{k+1} iteration to arrive at $(f^{F_{k+1}}(X, u), u + F_{k+1}w)$, and then the rest F_k iterations with the result

$$f^{F_{k+2}}(X, u) = f^{F_k}(f^{F_{k+1}}(X, u), u + wF_{k+1}). \quad (18)$$

To have a reasonable limit behavior of the sequence of the evolution operators we change scales for X and u by some appropriate factors α and β at each new step of the construction, and define the renormalized functions as

$$g_k(X, u) = \alpha^k f(X/\alpha^k, (-w)^k u). \quad (19)$$

Note that $wF_{k+1} = -(-w)^{k+1} \pmod{1}$, so it is natural to set $\beta = -1/w$. Rewriting (18) in terms of the renormalized functions we come to the functional equation

$$g_{k+2}(X, u) = \alpha^2 g_k(\alpha^{-1} g_{k+1}(X/\alpha, -wu), w^2 u + w). \quad (20)$$

The same equation was obtained in the RG analysis of the critical points TDT and TCT [15,16]. Here we will deal with some other solution of that equation, associated with the FT critical point. To find out of what kind this solution is, we may attempt to compute the functions g_k from direct iterations of the map (17).

As mentioned, the variable x in the original map must be distinguished from X used in the derivation of the RG equation. In other words, we have to produce a variable change to pass to an appropriate 'scaling coordinate system' in the two-dimensional phase space (u, x) . The new coordinates may be defined as

$$X \propto x - x_c + Pu + Qu^2, \quad U = u, \quad (21)$$

where x_c is obtained from Eq.(16) with substitution $\epsilon = \epsilon_c = 2$, $b = b_c$, P and Q are some coefficients.

To evaluate the coefficients P and Q we can act as follows. Let us perform iterations of the map (17) at the critical point $\epsilon = 2, b = b_c$, starting from $u_0 = 0$ and $x_0 = x_c$ (see (16)), and compute the values of u and x after F_k and F_{k+1} iterations. Let them be (u_{F_k}, x_{F_k}) and $(u_{F_{k+1}}, x_{F_{k+1}})$, respectively. Three points $(0, x_c)$, (u_{F_k}, x_{F_k}) , and $(u_{F_{k+1}}, x_{F_{k+1}})$ determine a parabola on the (u, x) -plane, and its equation is given by $x - x_c + Pu + Qu^2 = 0$. The coefficients may be easily evaluated from coordinates of the three points. Of course, the result will depend on the level number k , and we must estimate the asymptotical limits for the coefficients; they are $P = 5.92667$ and $Q = -210.629$. (In fact, the convergence is rather slow, but it is possible to guess its character, and obtain sufficiently good estimates.)

Now the procedure consists in the following:

- Fix k and the respective F_k .
- For given X and U define the initial conditions for the map (17): $x = XA\alpha^{-k} + x_c - PU - QU^2$, $u = U$ where A is an arbitrary constant, $\beta = -1/w$, and $\alpha = 2.89$ (this value has been selected in a course of the computations as the most appropriate one).
- Produce F_k iterations of the map (17).
- Return to variables (X, U) by the inverse change $X = \alpha^k A^{-1} (x_{F_k} - x_c + Pu_{F_k} + Qu_{F_k}^2)$, $U = u_{F_k}$.

Figure 6 presents graphically a sample of results of such computations for two Fibonacci numbers, $F_k = 233$ and 377. The 3D plots of two obtained functions are superimposed; observe their excellent agreement. (Yet better degree of coincidence was found for larger Fibonacci numbers.) This is an indication that we deal with a fixed-point solution of the functional equation

$$g(X, u) = \alpha^2 g(\alpha^{-1} g(X/\alpha, -wu), w^2 u + w). \quad (22)$$

Now it is worth emphasizing that the maps determining evolution over the Fibonacci numbers of iterations are constructed by a repetitive application of the fractional-linear mappings, and, hence, must relate to the same fractional-linear class. It implies that we may search for solution of the Eq.(20) in a form.

$$g_k(X, u) = \frac{a_k(u)X + b_k(u)}{c_k(u)X + d_k(u)}, \quad (23)$$

where the coefficients a, b, c, d are some functions of u . Without loss of generality we may require them to satisfy to an additional condition ('unimodularity')

$$a_k(u)d_k(u) - b_k(u)c_k(u) \equiv 1, \quad (24)$$

and set, as convenient, $c_k(0) = -1$. Substituting (23) into (20) we arrive at the RG equation reformulated in terms of the coefficients

$$\begin{pmatrix} a_{k+2}(u) & b_{k+2}(u) \\ c_{k+2}(u) & d_{k+2}(u) \end{pmatrix} = \begin{pmatrix} a_k(w^2 u + w) & \alpha^2 b_k(w^2 u + w) \\ c_k(w^2 u + w)/\alpha^2 & d_k(w^2 u + w) \end{pmatrix} \begin{pmatrix} a_{k+1}(-wu) & \alpha b_{k+1}(-wu) \\ c_{k+1}(-wu)/\alpha & d_{k+1}(-wu) \end{pmatrix}. \quad (25)$$

To find the fixed-point of this functional equation numerically we approximate the functions $a(u), b(u), c(u), d(u)$ by finite polynomial expansions. (Actually, the representation via Chebyshev's polynomials on an interval $u \in (-1, 1)$ has been used.) Then, we organize the RG transformation as a computer program, which calculates the set of the expansion coefficients for the functions $a_{k+2}(u), b_{k+2}(u), c_{k+2}(u)$ from two previous sets, $a_{k+1}(u), b_{k+1}(u), c_{k+1}(u)$ and $a_k(u), b_k(u), c_k(u)$. (Note that due to the unimodularity, only three of the four functions are independent.) The fixed-point conditions are

$$(a_{k+2}, b_{k+2}, c_{k+2}) = (a_{k+1}, b_{k+1}, c_{k+1}) \text{ and } (a_{k+1}, b_{k+1}, c_{k+1}) = (a_k, b_k, c_k). \quad (26)$$

In terms of the polynomial representation it is equivalent to some finite set of algebraic equations in respect to the unknown coefficients of the polynomials and the unknown constant α . This problem was solved by means of the multidimensional Newton method. As an initial guess, a function obtained from iterations of the original map (see Fig.6) was used. The resulting coefficients for the functions $a(u), b(u), c(u), d(u)$ corresponding to the fixed point are presented in Table II, and graphically in Fig.7(a). Figure 7(b) shows 3D plot for the fixed-point function; it may be compared with Fig.6. The constant α that is the scaling factor for X variable, is found to be

$$\alpha = 2.890\,053\,525\dots \quad (27)$$

in good agreement with the previously mentioned empirical estimate $\simeq 2.89$.

It is worth mentioning one more universal constant associated with the critical point.

Evaluating a derivative of $g(X, u) = (a(u)X + b(u))/(c(u)X + d(u))$ in respect to X at the origin yields $\gamma = [\partial g(X, u)/\partial X]_{X=0, u=0} = 1/d(0)^2 = 22.518745\dots$

As $g(X, u)$ represents the asymptotic form of the evolution operator for Fibonacci's numbers of iterations, and X differs from x only by the u -dependent shift, we conclude that the constant γ will appear as asymptotic value of the derivative $\partial x_{F_k}/\partial x_0$ if x_0 is selected in accordance with (17).

It gives a foundation for a method of locating the critical point. One composes a program to iterate the original map together with the recursive computation of $\partial x_n/\partial x_0$, and tries to select an appropriate value of b to obtain $\partial x_{F_k}/\partial x_0 = \gamma$. The result quickly converges to the critical point b_c as k grows. This method appears to be the most accurate, and the best numerical data (see (5)) have been obtained with its help.

6 Scaling properties of dynamics at the critical point

Let us consider attractor at the critical point of our model map (17). Its portrait is shown in Fig.8 in natural variables (u_n, x_n) . Depicting a part of the plot in 'scaling coordinates' (U, X) we reveal a scaling property intrinsic to the attractor: The structure is reproduced again and again at each subsequent step of magnification by factors α and β along the vertical and horizontal axes, respectively. This scaling property follows directly from the fact that in scaling coordinates the evolution operators for different Fibonacci numbers of iterations are asymptotically the same, up to the scale change (recall Fig.6).

From the scaling property one can deduce an asymptotic expression for the form of the invariant curve in small scales near the origin. The form reproduces itself under simultaneous scale change by factors α and $\beta = -1/w$ along the axes X and U , respectively, so, it must behave locally as $X \propto |U|^\kappa$ with $\kappa = \log \alpha / \log |\beta| \approx 2.2054$. As follows, this is a smooth curve, twice differentiable at the origin, but the third derivative diverges. Due to ergodicity ensured by irrationality of the frequency, the weak singularity at the origin implies existence of the same type of singularities over the whole invariant curve, on a dense set of points. Apparently, the observed wrinkled form of the invariant curve at the critical point (see Figs.2 and 8) reflects a presence of the mentioned set of dense weak singularities.

Figure 9 shows evolution of Fourier spectra generated by the map (17) as we move in the parameter plane along the bifurcation curve that corresponds to a threshold of intermittency. These spectra may be useful for a comparison with possible experimental studies of the transition. For small ϵ , i.e. far enough from the critical value, the spectrum contains a few components. It is enriched by many additional lines at intermediate frequencies as we come to the critical or supercritical values of ϵ . At the critical point the spectrum has a self-similar structure. It can be revealed by the use of the double logarithmic scale (as suggested in a different context in Refs.[10-12]), see Fig.10.

7 Linearized RG equation and spectrum of eigenvalues

A shift of parameters in the map (17) from the critical point corresponds to some perturbation of the evolution operator, and this perturbation will evolve under subsequent application of the RG transformation (20). Let us assume that the perturbation retains our evolution operators in the class of fractional-linear mappings. It means that we can search for solution of Eq.(20) in a form

$$g_k(X, u) = \frac{(a_k(u) + \tilde{a}_k(u))X + b_k(u) + \tilde{b}_k(u)}{(c_k(u) + \tilde{c}_k(u))X + d_k(u) + \tilde{d}_k(u)}, \quad (28)$$

where $a(u)$, $b(u)$, $c(u)$, $d(u)$ correspond to the fixed-point solution, while the terms with tilde are responsible for the perturbation. Then, a substitution $(\tilde{a}_k(u), \tilde{b}_k(u), \tilde{c}_k(u), \tilde{d}_k(u)) \propto \delta^k$ gives rise to the eigenvalue problem

$$\delta^2 \begin{pmatrix} \tilde{a}(u) & \tilde{b}(u) \\ \tilde{c}(u) & \tilde{d}(u) \end{pmatrix} = \delta \begin{pmatrix} a(w^2u + w) & \alpha^2 b(w^2u + w) \\ c(w^2u + w)/\alpha^2 & d(w^2u + w) \end{pmatrix} \begin{pmatrix} \tilde{a}(-wu) & \alpha \tilde{b}(-wu) \\ \tilde{c}(-wu)/\alpha & \tilde{d}(-wu) \end{pmatrix} + \begin{pmatrix} \tilde{a}(w^2u + w) & \alpha^2 \tilde{b}(w^2u + w) \\ \tilde{c}(w^2u + w)/\alpha^2 & \tilde{d}(w^2u + w) \end{pmatrix} \begin{pmatrix} a(-wu) & \alpha b(-wu) \\ c(-wu)/\alpha & d(-wu) \end{pmatrix}. \quad (29)$$

As usual, only the eigenvalues larger than 1 in modulus may be of interest because the respective perturbations grow under repetition of the RG transformation and, hence, may influence the form of the long-time evolution operators.

Numerically, we solved the problem by use of finite polynomial approximations for the functions involved, with taking into account the previously found fixed-point solution. The equation (29) then gives rise to an eigenvalue problem defined in terms of finite-dimensional matrices acting in a vector space of coefficients for the polynomial expansions of the functions $\tilde{a}(u)$, $\tilde{b}(u)$, $\tilde{c}(u)$, $\tilde{d}(u)$, and it can be dealt with standard methods of linear algebra. The computations reveal 13 eigenvalues larger or equal to 1 in modulus; they are listed in Table III.

Actually, only few of them are of relevance.

First, some of the found eigenvectors are associated with infinitesimal variable changes. For example, with a substitution $X \rightarrow X + \varepsilon$ in the map (23) we arrive at a new map

$$X_{n+1} = \frac{a(u)X_n + \varepsilon a(u) + b(u)}{c(u)X_n + \varepsilon c(u) + d(u)} - \varepsilon. \quad (30)$$

The right-hand part is represented in the first order in ε as

$$g(X, u) + \tilde{g}(X, u) \cong \frac{(a(u) + \tilde{a}(u))X_n + (b(u) + \tilde{b}(u))}{(c(u) + \tilde{c}(u))X_n + (d(u) + \tilde{d}(u))}, \quad (31)$$

where $(\tilde{a}(u), \tilde{b}(u), \tilde{c}(u), \tilde{d}(u)) = \varepsilon(1 - c(u), -d(u), 1, 0)$. In a course of the RG transformation X is rescaled as $X \rightarrow X/\alpha$, and, respectively, the renormalized relative coordinate shift is multiplied by α too. So, $(1 - c(u), -d(u), 1, 0)$ represents an eigenvector, and α is the associated eigenvalue. (One can verify it by a direct substitution of the eigenvector into Eq.(29); moreover, the assertion has been checked also by accurate comparison of the present eigenvector components with those found numerically as functions of u .) So, we may exclude the eigenvalue α from the list, because a perturbation of this kind in the evolution operator may be always compensated by a shift of the origin. In the same way, we regard several other eigenvectors as irrelevant, they are marked in Table III as linked with variable changes.

Second, in the derivation of the RG equation we have used a definite order for the evolution operators (first F_{k+1} , then F_k steps). However, this order may be inverted, and it leads to a distinct alternative formulation of the eigenvalue problem, namely,

$$\delta^2 \begin{pmatrix} \tilde{a}(u) & \tilde{b}(u) \\ \tilde{c}(u) & \tilde{d}(u) \end{pmatrix} = \delta \begin{pmatrix} \tilde{a}(-wu + w^{-1}) & \alpha \tilde{b}(-wu + w^{-1}) \\ \tilde{c}(-wu + w^{-1})/\alpha & \tilde{d}(-wu + w^{-1}) \end{pmatrix} \begin{pmatrix} a(w^2u) & \alpha^2 b(w^2u) \\ c(w^2u)/\alpha^2 & d(w^2u) \end{pmatrix} + \begin{pmatrix} a(-wu + w^{-1}) & \alpha b(-wu + w^{-1}) \\ c(-wu + w^{-1})/\alpha & d(-wu + w^{-1}) \end{pmatrix} \begin{pmatrix} \tilde{a}(w^2u) & \alpha^2 \tilde{b}(w^2u) \\ \tilde{c}(w^2u)/\alpha^2 & \tilde{d}(w^2u) \end{pmatrix}. \quad (32)$$

The true evolution operators are in any case the multiple compositions of the same original map. Thus, for the actual perturbations the equations (29) and (32) must be equivalent. In other words, only those eigenvectors may be of relevance, which are common for both the eigenvalue problems. This property was verified numerically for all found eigenvectors. As observed, some of them do not satisfy the condition; in Table III they are marked as relating to a non-commutative subspace.

The rest two eigenvalues

$$\delta_1 = 3.134\,272\,989\dots \text{ and } \delta_2 = w^{-1} = 1.618\,033\,979\dots \quad (33)$$

are relevant and responsible for scaling properties of the parameter space near the critical point TF.

If we depart from the critical point in the parameter plane along the bifurcation curve of the attractor-repeller collision, the first eigenvector appears not to contribute into the perturbation of the evolution operator. In this case the only relevant perturbation is associated with δ_2 . However, if we choose a transversal direction, say, along the axis b , the perturbation of the first kind appears. It means that a coordinate system appropriate for observation of scaling in the parameter plane has to be defined as shown in Fig.11. It is a curvilinear system: one coordinate axis is the line $\epsilon = 2$, but another follows the bifurcation border accounting its curvature. In analytical expression it is sufficient to keep terms up to the second order. (This is due to the concrete relation between δ_1 and δ_2 : $\delta_1 > \delta_2$ and $\delta_1 > \delta_2^2$, but $\delta_1 < \delta_2^3$; see other examples of scaling coordinates for different critical points and discussion of the role of the relation of the eigenvalues in Refs [13,16,30,31].)

So, we set

$$\begin{aligned} b &= b_c + C_1 + pC_2 + qC_2^2, \\ \epsilon &= 2 + C_2, \end{aligned} \quad (34)$$

where

$$p = (2 - b_c)/4 \cong -0.64938, \quad q \cong -0.33692. \quad (35)$$

The expression for p follows from the analogy with the Harper equation and from the Aubry transformation rule (15): An infinitesimal shift of ϵ and b along the tangent line to the bifurcation border must correspond to

the shift of ϵ' and b' along the same line. The value of q is calculated numerically, from the curvature of the bifurcation border.

Beside the obtained nontrivial solution of the RG equation there exists also a trivial, phase independent fixed point

$$g(X, u) \equiv g(X) = X/(1 - X), \quad (36)$$

with $\alpha = 1/w = 1.618034\dots$. Naturally, this is the fixed point responsible for behavior on the subcritical part of the bifurcation border and associated with the transition accompanied by a collision of smooth invariant curves. The eigenvalue problem for the linearized RG equation may be solved analytically for this case, and it reveals a unique relevant eigenvalue $\delta = 1/w^2 = 2.618034\dots$

8 Dynamics in a neighborhood of the critical point and intermittency

Let us discuss now a question on the peculiarities of intermittency in the quasiperiodically forced map. First of all, we outline a possibility of three distinct regimes at the onset of intermittency:

- subcritical, $\epsilon < \epsilon_c = 2$, when collision with coincidence of the smooth invariant curves (attractor and repeller) takes place at the moment of bifurcation;
- critical, $\epsilon = \epsilon_c$, it corresponds to collision and coincidence of the wrinkled invariant curves (the threshold of fractalization);
- supercritical, $\epsilon > \epsilon_c$, there collision of the invariant curves occurs at some fractal subset of points.

Figure 12 (a)-(c) show the time dependencies for dynamical variable generated by the model map (1) just before and after the transition for the mentioned three cases. In the intermittent regimes the 'laminar stages' are interrupted by the 'turbulent bursts'. The laminar stages on the right panels reproduce approximately the patterns of the left panels.

Relative duration of the laminar phases becomes larger as we approach the transition point. In usual Pomeau - Manneville intermittency of type I the average duration of the laminar stages behaves as

$$t_{\text{lam}} \propto 1/|\Delta b|^\nu \quad (37)$$

with $\nu = 0.5$ [17-19].

In presence of the quasiperiodic force the same law is valid in the subcritical region $\epsilon < 2$. In the critical case $\epsilon = 2$ the exponent is distinct. Indeed, as follows from the RG results, to observe increase of a characteristic time scale by factor $\theta = 1/w = 1.61803\dots$ we have to decrease a shift of parameter b from the bifurcation threshold by factor $\delta_1 = 3.13427\dots$. As follows, the exponent must be

$$\nu = \log \theta / \log \delta_1 \cong 0.42123. \quad (38)$$

(Note that substitution of δ factor associated with the trivial fixed point (36) yields just the result for the subcritical region, $\nu = 0.5$.)

Figure 13 shows the data of numerical experiments with the fractional-linear map aimed to verify the theoretical predictions for the exponent ν . At each fixed ϵ we empirically determined an average duration of passage through the 'channel' in dependence on Δb for ensemble of orbits with random initial conditions and plotted the results in the double logarithmic scale. For particular $\epsilon = 1.7$ (subcritical) and 2 (critical) the dependencies fit the straight lines of a definite slope. As seen from Table IV, the correspondence of numerical results with the theoretical predictions is rather good. At subcritical ϵ slightly less than 2 one can observe a 'crossover' phenomenon, that is the slope change from critical to subcritical value at some intermediate value of Δb .

It is interesting that the results obtained for the supercritical region also indicate a constant definite slope, $\nu \approx 0.45$. At this moment it is not clear how to explain this observation theoretically.

Figure 14 shows diagrams for Lyapunov exponents versus parameter b for subcritical, critical, and supercritical constant values of ϵ in the artificial map (1), which includes the reinjection mechanism.

For the subcritical case the intermittency threshold corresponds to the onset of chaos: The Lyapunov exponent becomes positive immediately after the transition.

In the supercritical region the Lyapunov exponent is yet negative at the moment of the bifurcation, it cannot immediately become positive, and the transition will be accompanied by a birth of an SNA rather than a chaotic attractor.

In the critical situation the diagram demonstrates self-similarity (at least, in the domain $b < b_c$): A magnification by factor $\delta_1 = 3.13427 \dots$ along the horizontal axis, and by factor $\theta = 1.61803 \dots$ (the rescaling factor for time) along the vertical axis gives rise to similar pictures.

Figure 15 demonstrates a chart of parameter plane, or the 'phase diagram' in the natural variables (left panel) and in the scaling coordinates (the right panel). The gray areas are those of negative Lyapunov exponent, two distinct tones correspond to the domains of existence of the localized attractors (smooth tori in the bottom area) and to the intermittent regimes (the top area to the right). Apparently, the last is the region of SNA. This assertion may be deduced from arguments of Pikovsky and Feudel [6]. Indeed, considering dynamics there in terms of rational approximants one can notice that the phase-dependent bifurcations will occur inevitably. In contrast, the white area of positive Lyapunov exponent is the domain of chaotic intermittent regimes. Figure 16 shows portraits of attractors at several representative points of the phase diagram.

9 Conclusion

The present study was devoted to one special situation of transition from conventional quasiperiodicity ('smooth torus') to chaos or SNA via intermittency in a model map under quasiperiodic external driving with frequency parameter defined as the inverse golden mean. The main attention was concentrated on a critical situation reached at one particular, sufficiently large amplitude of driving, associated with threshold of fractalization. Here the bifurcation transition analogous to the tangent bifurcation consists in collision with coincidence and subsequent disappearance of an attractor and a repeller represented by a pair of wrinkled invariant curves. RG analysis appropriate for the critical situation was developed, the fixed-point solution of the RG equation was found in a class of fractional-linear functions, the constants responsible for scaling in phase space and parameter space were computed.

Some related problems remain yet open, for example, concerning global scaling properties and dimensions of the critical attractor. Also a generalization for other irrational rotational numbers is of interest. (The last seems undoubtedly possible because of the analogy with Harper equation: there the criticality at $\epsilon = 2$ occurs at arbitrary values of the rotational number.) One more problem is a development of an appropriate approach to analysis of the transition in supercritical region, which would be of the same significance, as the RG method is in the critical and subcritical cases.

As is common in situations allowing the RG analysis, one can expect that the quantitative regularities intrinsic to our model map will be valid also in other systems relating to the same universality class. In particular, it may be suggested that transition to SNA observed in a quasiperiodically forced subcritical circle map [28] is of the same nature. Also, it would be significant to find this type of behavior in systems of higher dimension, for example, in quasiperiodically driven invertible 2D maps, which could represent Poincare maps of some flow systems.

It would be interesting to reveal details and regularities of coexistence and subordination of the discussed type of critical behavior with behaviors of distinct universality classes studied in Refs. [14-16] (e.g. the torus-collision terminal and torus-doubling terminal points).

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References

- [1] L.D.Landau, Doklady Acad. Sci. URSS **44**, 311 (1944).
- [2] E.Hopf, Comm. Appl. Math. **1**, 303 (1948).
- [3] D.Ruelle and F.Takens, Comm. Math. Phys. **20**, 167 (1971).
- [4] C.Grebogi, E.Ott, S.Pelikan and J.A.Yorke, Physica **D13**, 261 (1984).
- [5] F.J.Romeiras, A.Bondeson, E.Ott, T.M.Antonsen, and C.Grebogi, Physica **D26**, 277 (1987).
- [6] A.S.Pikovsky and U.Feudel, Chaos **5**, 253 (1995).
- [7] M.J.Feigenbaum, J. Statist. Phys. **19**, 25 (1978).
- [8] M.J.Feigenbaum, J. Statist. Phys. **21**, 669 (1979).
- [9] E.B.Vul, Y.G.Sinai, and K.M.Khanin, Russ. Math. Surv. **39**(3), 1 (1984).
- [10] D.Rand, S.Ostlund, J.Sethna, and E.D.Siggia, Phys. Rev. Lett. **49**, 132 (1982).
- [11] S.Ostlund, D.Rand, J.Sethna, and E.D.Siggia, Physica **D8**, 303 (1983).
- [12] M.J.Feigenbaum, L.P.Kadanoff, S.J.Shenker, Physica **D5**, 370 (1982).
- [13] A.P.Kuznetsov, S.P.Kuznetsov, and I.R.Sataev, Physica **D109**, 91 (1997).
- [14] S.P.Kuznetsov, A.S.Pikovsky, and U.Feudel, Phys. Rev. **E51**, R1629 (1995).
- [15] S.Kuznetsov, U.Feudel, and A.Pikovsky, Phys. Rev. **E57**, 1585 (1998).
- [16] S.P.Kuznetsov, E.Neumann, A.Pikovsky, and I.R.Sataev, Phys. Rev. **E62**, 1995 (2000).
- [17] Y.Pomeau and P.Manneville, Comm. Math. Phys. **74**, 189 (1980).
- [18] B. Hu and J. Rudnick, Phys. Rev. Lett. **48**, 1645 (1982).
- [19] J.E.Hirsch, B.A.Huberman, and D.J.Scalapino, Phys. Rev. **A25**, 519 (1982).
- [20] F.Argoul and A.Arneodo, J. Phys. Lett. Paris **46**, L901 (1985).
- [21] U.Feudel, A.S.Pikovsky, and J.Kurths, Physica **D88**, 176 (1995).
- [22] P.G.Harper, Proc. Phys. Soc. London, **A68**, 854 (1955).
- [23] I.M.Suslov, Zh. Exp. Theor. Fiz. **83**, 1079 (1982).
- [24] S.Ostlund and R.Pandit, Phys. Rev. **B29**, 1394 (1984).
- [25] J.A.Ketoja and I.I.Satija, Physica **D109**, 70 (1997).
- [26] B.D.Mestel, A.H.Osbaldestin, and B.Winn, J. Math. Phys. **41**, 8304 (2000).
- [27] S.S.Negi and R.Ramaswamy, Phys. Rev. **E64**, 045204 (2001).
- [28] H.Osinga, J.Wiersig, P.Glendinning and U.Feudel, Int. J. Bifurcation Chaos Appl. Sci. Eng., **11**, 3085 (2001).
- [29] S.Aubry and G.Andre, in *Proceedings of the Israel Physical Society*, ed. C.G.Kuper (Hilger, Bristol, 1979), Vol.3, p.133.
- [30] A.P.Kuznetsov, S.P.Kuznetsov, and I.R.Sataev, Phys. Lett. **A189**, 367 (1994).
- [31] N.Y.Ivankov and S.P.Kuznetsov, Phys. Rev. **E63**, 046210 (2001).

Figure captions

Figure 1. Onset of chaos via the Pomeau-Manneville intermittency of type I in the model map (1) at $\epsilon = 0$ (a), and the transition via collision and disappearance of smooth closed invariant curves, attractor and repeller (b). The diagrams are shown on the plane (x, x') , where x' relates to the moment of time one unit later than x . The left panels correspond to situation before the bifurcation, and the right panels to situation after the transition.

Figure 2. Bifurcation of collision of the closed invariant curves in subcritical (a), critical (b), and supercritical (c) situations, at $\epsilon = 1.7, 2$, and 2.3 , respectively. The invariant curves representing attractor are shown by solid lines, and those for repeller by dashed. Parameter b grows from the left to the right, and the last panel in a row corresponds to the moment of collision.

Figure 3. The Lyapunov exponent at the bifurcation border versus the amplitude parameter ϵ (a) and dependence of the Lyapunov exponent on parameter b at critical $\epsilon = 2$ (b) for the fractional-linear map with quasiperiodic driving.

Figure 4. Floquet eigenvalue, or multiplier μ computed at three subsequent levels of rational approximation versus phase variable u at the values of b corresponding to the moments of the first cycle collision for subcritical (a), critical (b), and supercritical (c) amplitudes.

Figure 5. Transmission bands (gray) and forbidden zones (white) in the parameter plane (ϵ, Ω) for the Harper equation (a), and the chart of dynamical regimes (phase diagram) for intermittency under quasiperiodic driving (b). The bifurcation curve of the attractor-repeller collision on the panel (b) exactly corresponds to the bottom border of the gray area in the diagram (a).

Figure 6. 3D plots for the functions obtained from direct iterations of the fractional-linear map at $\epsilon = 2$, $b = b_c = -0.597515\dots$ with appropriate renormalization as explained in text (the arbitrary constant mentioned in the text is chosen as $A = 84$). The number of iterations $F_k = 233$ and 377 . A coincidence of both plots indicate that the functions approach the fixed point of the RG transformation.

Figure 7. Coefficients for the fixed-point fractional-linear solution of the RG equation versus phase variable u (a) and 3D plot of the universal function (b).

Figure 8. Portrait of the critical attractor in natural variables $u_n = nw + u$, x_n (left panel) and fragments of the picture shown under subsequent magnification in the curvilinear 'scaling coordinate' system. At each next step the magnification is increased by factor $\alpha = 2.89005\dots$ along the vertical axis and $\beta = -1.61803\dots$ along the horizontal axis.

Figure 9. Evolution of Fourier spectra generated by the map (3) along the bifurcation curve that corresponds to a threshold of intermittency.

Figure 10. Fourier spectrum at the TF critical point presented in the double logarithmic scale. Notice visible repetition of the structure in respect to a shift along the frequency axis.

Figure 11. Local coordinates on the parameter plane of the fractional-linear map appropriate for demonstrating of scaling.

Figure 12. The dynamical variable versus time in the model map (1) just before and after the transition: (a) a subcritical amplitude of driving, $\epsilon = 0.5$; (b) the critical case, $\epsilon = 2$; (c) a supercritical case, $\epsilon = 2.3$.

Figure 13. Data of numerical experiments with the fractional-linear map: average duration of passage through the 'channel' versus deflection from the bifurcation threshold for several values of ϵ in the double logarithmic scale. Observe a 'crossover' phenomenon, the slope change from critical to subcritical value at some intermediate value of Δb for $\epsilon = 1.95$.

Figure 14. Lyapunov exponent versus parameter b for subcritical, critical, and supercritical constant values of ϵ in the map (1). Illustration of scaling for the critical case: insets are shown with subsequent magnification by $\delta_1 = 3.13427\dots$ along the horizontal axis, and by factor $\theta = 1.61803\dots$ (the rescaling factor for time) along the vertical axis.

Figure 15. A chart of parameter plane or the 'phase diagram' in natural variables (left panel) and in

scaling coordinates (the right panel). The gray areas correspond to negative Lyapunov exponent values, with distinct tones for localized attractors – smooth tori in the bottom area, and to intermittent regimes associated presumably with SNA in the top area to the right.

Figure 16. Portraits of attractors at several representative points of the phase diagram.

Table 1: The values of b for the first attractor-repeller collision at the critical amplitude $\varepsilon = 2$

w_k	b
8/13	-0.5989496730498198
13/21	-0.5993564164969890
21/34	-0.5975700101088623
34/55	-0.5977371349948819
55/89	-0.5975077597293093
89/144	-0.5975427315192966
144/233	-0.5975125430279679
233/377	-0.5975187094612125
377/610	-0.5975146415227064
610/987	-0.5975156490874847
987/1597	-0.5975150899217102
1597/2584	-0.5975152478940331
2584/4181	-0.5975151698106684
4181/6765	-0.5975151939749183
6765/10946	-0.5975151829388339
10946/17711	-0.5975151865779841

Table 2: The polynomial coefficients for the fixed-point solution of the RG equation

	$a(u)$	$b(u)$	$c(u)$	$d(u)$
1	3.180070169	0.329861441	-1.000000000	0.210730746
u	-3.450688327	-0.003027254	-0.601533776	0.167219763
u^2	-3.247090086	-8.330520346	0.443802007	3.062831633
u^3	3.992032627	9.444480715	0.293604279	1.865595582
u^4	1.200194704	11.133767028	-0.094027530	-1.597931152
u^5	-1.549680177	-13.205541847	-0.058193081	-1.044745269
u^6	-0.187890799	-4.313300571	0.011160724	0.358377518
u^7	0.286983991	5.592822816	0.006427031	0.223393531
u^8	0.016720852	0.734433497	-0.000847235	-0.044603872
u^9	-0.031115111	-1.104258820	-0.000457454	-0.025974096
u^{10}	-0.000950352	-0.069739548	0.000044648	0.003533563
u^{11}	0.002215271	0.125757870	0.000022730	0.001923598
u^{12}	0.000035770	0.004131194	-0.000001582	-0.000192702
u^{13}	-0.000110612	-0.009312156	-0.000000761	-0.000098574
u^{14}	-0.000000838	-0.000160051		0.000006910
u^{15}	0.000003663	0.000479736		0.000003335
u^{16}		0.000003822		
u^{17}		-0.000016158		

Table 3: The eigenvalues larger or equal to 1 in modulus

Eigenvalue	Designation	Interpretation
3.134272989	δ_1	Relevant eigenvalue
-2.890053625	$-\alpha$	Non-commutative subspace
2.890053625	α	The variable change $x \leftarrow x + \varepsilon$
-1.786151370	$-\alpha w^{-1}$	The variable change $x \leftarrow x + \varepsilon u$
1.786151370	αw^{-1}	Non-commutative subspace
-1.618033979	$-w^{-1}$	The variable change $u \leftarrow u + \varepsilon$
1.618033979	$\delta_2 = w^{-1}$	Relevant eigenvalue
1.618033979	w^{-1}	Violation of the unimodularity
1.103902257	αw^{-2}	The variable change $x \leftarrow x + \varepsilon u^2$
-1.103902257	$-\alpha w^{-2}$	Non-commutative subspace
1.000000000	1	The variable change $x \leftarrow x(1 + \varepsilon)$
-1.000000000	-1	Non-commutative subspace
-1.000000000	-1	Non-commutative subspace

Table 4: Comparison of the numerical results and RG predictions for the critical exponent

ε	ν , numerics	ν , theory
1.7	0.508	0.5
2.0	0.424	0.42123
2.2	0.452	?
2.3	0.456	?

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